Int J Theor Phys (2008) 47: 741–750
DOI 10.1007/s10773-007-9498-8

Soliton Fusion and Fission Phenomena in the (2 + 1)-Dimensional Variable Coefficient Broer-Kaup System

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Received: 15 May 2007 / Accepted: 16 July 2007 / Published online: 9 August 2007

Abstract In this paper, the general projective Riccati equation method is applied to derive variable separation solutions of the (2 + 1)-dimensional variable coefficient Broer-Kaup system. By further studying, we find that these variable separation solutions obtained by PREM, which seem independent, actually depend on each other. Based on the variable separation solution and choosing suitable functions p and q, new types of fusion and fission phenomena among bell-like semi-foldons are firstly investigated.

Keywords General projective riccati equation method · (2 + 1)-dimensional variable coefficient BK system · Soliton fusion and fission phenomena

1 Introduction

It is well known that the interaction of soliton in (1 + 1)-dimensional nonlinear models are usually considered to be elastic. That means there is no exchange of any physical quantities like the energy and the momentum among interacting solitons. That is, except for the phase shifts, the shape and velocities of soliton all remain unchanged. However, for some (1 + 1)-dimensional models, two or more solitons may fuse into one soliton at a special time while sometimes one soliton may fission into two or more solitons at other special times [1]. These phenomena are often called soliton fusion and soliton fission, respectively. Actually, the soliton fusion and fission phenomena have been observed in many physical systems, such as organic membrane and macromolecular material [2], and physical fields, like plasma physics, nuclear physics and hydrodynamics [3]. Recently, Zhang et al. [4] and Lin et al. [5] studied the evolutions of soliton solutions for two (1 + 1)-dimensional PDEs with time and revealed the soliton fission and soliton fusion phenomena. In a similar way, Wang et al. [6] (and references therein) further discussed two (1 + 1)-dimensional models, the Burgers equation and the Sharma-Tasso-Olver equation, via Hirotas direct method, and

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also found the soliton fission and soliton fusion phenomena. Zheng et al. [7] revealed special soliton fission and fusion phenomena in the $(2+1)$-dimensional Broer-Kaup-Kupershmidt system. However, to the best of our knowledge, very few reports on soliton fusion and fission behaviors in higher-dimensional nonlinear physical models except [7] have been found, especially, in some well-known nonlinear dynamic systems. The fusion and fission phenomena among higher dimensional multi-valued coherent localized structures, such as semifoldons, are hardly discussed. Motivated by this reason, we will report and discuss these phenomena in the following well-known $(2+1)$-dimensional variable coefficient Broer-Kaup (VCBK) system

$$\begin{aligned} &H_{ty} - B(t)[H_{xxy} - 2(HH_x)_y - 2G_{xx}] = 0, \\ &G_t + B(t)[G_{xx} + 2(HG)_x] = 0, \end{aligned} \tag{1}$$

where $B(t)$ is an arbitrary function of time t. It is evident that when $B(t) = \mathit{constant}$, the VCBK system will be degenerated to the celebrated $(2+1)$-dimensional Broer-Kaup system (BK) [8], which may be derived from inner parameter dependent symmetry constraint of Kadomtsev-Petviashvili model [9]. When $y = x$, the $(2+1)$-dimensional BK system will be reduced further to the usual $(1+1)$-dimensional BK system, which can be used to describe the propagation of long waves in shallow water [10]. Using some suitable dependent and independent variable transformations, Chen and Li [11] have proved that the $(2+1)$-dimensional BK system can be transformed to the $(2+1)$-dimensional dispersive long wave equation (DLWE) [12] and $(2+1)$-dimensional Ablowitz-Kaup-Newell-Segur system (AKNS) [13].

Nowadays, many simple and effective direct methods, such as the extended tanh-function method (ETM) based on mapping method [14], the projective Riccati equation method (PREM) [15] and the Jacobian-function method [16], etc., are developed to derive travelling wave solutions of the nonlinear evolution equations. Recently, the ETM based on mapping method has been the alternative method to realize the variable separation for the $(2+1)$-dimensional Broer-Kaup-Kupershmidt (BKK) system [17], modified dispersive water-wave (MDWW) system [18], dispersive long wave (DLW) equation [19], generalized BK system [20], and so on. More recently, we obtain the variable separation solutions of a class of nonlinear evolutional equations by means of the ETM [21]. Now a natural and important issue is that whether the variable separation solutions based on the former ETM can be derived by other direct methods. The crucial question is how to obtain solutions with certain arbitrary functions.

In this paper, we successfully generalize the projective Riccati equation method (PREM) to obtain variable separation solutions of $(2+1)$-dimensional VCBK equation. By further studying, we find that these variable separation solutions obtained by PREM, which seem independent, actually depend on each other. We can see this fact from the $(2+1)$-dimensional VCBK system, which will be discussed as an example.

2 Review of the General Projective Riccati Equation Method

The basic idea of the general PREM is that: for a given nonlinear partial differential equation (NPDE) with independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$, and dependent variable u,

$$L(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{2}$$

where L is in general a polynomial function of its argument, and the subscripts denote the partial derivatives.

Step 1: one assumes that (2) possesses the following ansatz

Type I

$$u=a_0(x)+\sum_{i=1}^{l} f^{i-1}[w(x)]\{a_i(x)f[w(x)]+b_i(x)g[w(x)]\}, \tag{3}$$

where $a_0=a_0(x), a_i=a_i(x), b_i=b_i(x)$ $(i=1,\dots,l)$, $w=w(x)$ are all arbitrary functions of indicated variables, and $f(w), g(w)$ satisfy

$$f'(w)=\epsilon f(w)g(w), \qquad g'(w)=R+\epsilon g^2(w)-rf(w), \quad \epsilon=\pm 1, \tag{4}$$

$$g^2(w)=-\epsilon\left[R-2rf(w)+\frac{r^2+\mu}{R}f^2(w)\right], \quad R\neq 0, \mu=\pm 1, \tag{5}$$

where R, r are two constants and $'$ denotes $\frac{d}{dw}$. Or we seek solutions of (2) in the following form

Type II When $R=r=0$ in (4) and (5)

$$u=a_0(x)+\sum_{i=1}^{l} a_i(x)g^i[w(x)], \tag{6}$$

where $g(w)$ satisfies

$$g'(w)=g^2(w). \tag{7}$$

The parameter l is determined by balancing the highest order derivative terms with the nonlinear terms in (2).

Step 2: Substituting (3) along with (4) and (5) [or (6) along with (7)] into (2) yields a set of polynomials for $f^i g^j$ $(i=0,1,\dots,\ j=0,1)$. Eliminating all the coefficients of the powers of $f^i g^j$, yields a series of partial differential equations, from which the parameters a_0, a_i, b_i $(i=1,\dots,l)$ and w are explicitly determined.

Step 3: We know from [15, 22–24] that (4) and (5) admit the following solutions:

Case 1 When $\epsilon=-1, \mu=-1, R\neq 0$,

$$f_1(w)=\frac{R\operatorname{sech}(\sqrt{R}w)}{r\operatorname{sech}(\sqrt{R}w)+1}, \qquad g_1(w)=\frac{\sqrt{R}\tanh(\sqrt{R}w)}{r\operatorname{sech}(\sqrt{R}w)+1}. \tag{8}$$

Case 2 When $\epsilon=-1, \mu=1, R\neq 0$,

$$f_2(w)=\frac{R\operatorname{csch}(\sqrt{R}w)}{r\operatorname{csch}(\sqrt{R}w)+1}, \qquad g_2(w)=\frac{\sqrt{R}\coth(\sqrt{R}w)}{r\operatorname{csch}(\sqrt{R}w)+1}. \tag{9}$$

Case 3 When $\epsilon = 1, \mu = -1, R \neq 0$,

$$f_3(w) = \frac{R\sec(\sqrt{R}w)}{r\sec(\sqrt{R}w)+1}, \qquad g_3(w) = \frac{\sqrt{R}\tan(\sqrt{R}w)}{r\sec(\sqrt{R}w)+1}, \tag{10}$$

$$f_4(w) = \frac{R\csc(\sqrt{R}w)}{r\csc(\sqrt{R}w)+1}, \qquad g_4(w) = -\frac{\sqrt{R}\cot(\sqrt{R}w)}{r\csc(\sqrt{R}w)+1}. \tag{11}$$

Case 4 When $R = r = 0$,

$$f_5(w) = \frac{C}{w} = C\epsilon g_5(w), \qquad g_5(w) = \frac{1}{\epsilon w}, \tag{12}$$

where C is a constant. Substitute the parameters a_0, a_i, b_i $(i = 1, \ldots, l)$ and w obtained in Step 2 into (3) along with (4) and (5) [or (6) along with (7)] to obtain variable separation solutions of the NLPDE of concern.

Remark 1 When $\epsilon = -1, R = 1, \mu = \mu/k$, (2) becomes a projective Ricatti equation [15]. When $\epsilon = -1, R = 1$, (2) becomes the projective Ricatti equation, by means of which a $(2+1)$-dimensional simplified generalized Broer-Kaup (SGBK) system was studied in [22].

Remark 2 The PREM is firstly generalized to derive variable separation solutions of nonlinear evolutional equations in this paper. Actually, these solutions in (8–12), which seem independent, depend on each other if they are applied to find variable separation solution for nonlinear soliton systems. This fact can be concluded from the variable separation solutions of the $(2+1)$-dimensional VCBK equation, which will be discussed in detail next. In these solutions (8–12), only the solution (12) is essentially effective, while other solutions related to tan, cot, tanh and coth functions are special cases of (12).

3 Variable Separation Solutions for the $(2+1)$-Dimensional VCBK System

To solve the $(2+1)$-dimensional VCBK system, first, let us make a transformation for (1): $G = H_y$. Substituting the transformation into (1) yields

$$H_{ty} + B(t)[2(H_x H)_y + H_{xxy}] = 0. \tag{13}$$

Along with the general projective Riccati equation method, according to Step 2 in Sect. 2, by balancing the higher-order derivative term with the nonlinear term in (13), we get $l = 1$ in (3). Therefore we suppose that (13) has the following formal solution with $R \neq 0$,

$$H(x, y, t) = a_0(x, y, t) + a_1(x, y, t)f(w) + b_1(x, y, t)g(w), \tag{14}$$

where f and g satisfy (4) and (5) with (8–11) and $w \equiv w(x, y, t)$. Inserting (14) with (4) and (5) into (13), selecting the variable separation ansatz

$$w = p(x, t) + q(y), \tag{15}$$

and eliminating all the coefficients of polynomials of $f^i g^j$ $(i = 0 \sim 4; j = 0, 1)$, one gets a set of partial differential equations. It is very difficult to solve these prolix and complicated differential equations. Fortunately, by careful analysis and calculation, we derive the special solutions

$$a_0 = -\frac{B(t)p_{xx} + p_t}{2B(t)p_x}, \qquad a_1 = \frac{p_x}{2}\sqrt{\frac{-\epsilon(r^2+\mu)}{R}}, \qquad b_1 = -\frac{p_x}{2\epsilon}. \tag{16}$$

Therefore, the variable separation solutions of the $(2+1)$-dimensional VCBK equation are of form

Case 1 $\epsilon = -1, \mu = -1$,

$$\begin{aligned} H_1 &= -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{\frac{r^2-1}{R}}\frac{R\,\text{sech}[\sqrt{R}(p+q)]}{r\,\text{sech}[\sqrt{R}(p+q)]+1} + \frac{p_x}{2}\frac{\sqrt{R}\tanh[\sqrt{R}(p+q)]}{r\,\text{sech}[\sqrt{R}(p+q)]+1} \\ &\equiv -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{R}\tanh\left[\sqrt{R}\frac{(p+q)+\varphi_1}{2}\right], \end{aligned} \tag{17}$$

$$G_1 = H_{1y} = \frac{R}{4}p_x q_y\,\text{sech}^2\left[\sqrt{R}\frac{(p+q)+\varphi_1}{2}\right]. \tag{18}$$

In (17), we use the relation

$$\frac{\sqrt{r^2-1}\,\text{sech}(\theta) + \tanh(\theta)}{r\,\text{sech}(\theta)+1} \equiv \tanh\left(\frac{\theta+\varphi_1}{2}\right),$$

where

$$\varphi_1 = \tanh^{-1}\left(\frac{\sqrt{r^2-1}}{r}\right).$$

Case 2 $\epsilon = -1, \mu = 1$,

$$\begin{aligned} H_2 &= -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{\frac{r^2+1}{R}}\frac{R\,\text{csch}[\sqrt{R}(p+q)]}{r\,\text{csch}[\sqrt{R}(p+q)]+1} + \frac{p_x}{2}\frac{\sqrt{R}\coth[\sqrt{R}(p+q)]}{r\,\text{csch}[\sqrt{R}(p+q)]+1} \\ &\equiv -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{R}\coth\left[\sqrt{R}\frac{(p+q)+\varphi_2}{2}\right], \end{aligned} \tag{19}$$

$$G_2 = H_{2y} = -\frac{R}{4}p_x q_y\,\text{csch}^2\left[\sqrt{R}\frac{(p+q)+\varphi_2}{2}\right]. \tag{20}$$

In (19), we use the relation

$$\frac{\sqrt{r^2+1}\,\text{csch}(\theta) + \coth(\theta)}{r\,\text{csch}(\theta)+1} \equiv \coth\left(\frac{\theta+\varphi_2}{2}\right),$$

where

$$\varphi_2 = \tanh^{-1}\left(\frac{r}{\sqrt{r^2+1}}\right).$$

Case 3 $\epsilon = 1, \mu = -1$,

$$H_3 = -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{\frac{1-r^2}{R}}\frac{R\sec[\sqrt{R}(p+q)]}{r\sec[\sqrt{R}(p+q)]+1} - \frac{p_x}{2}\frac{\sqrt{R}\tan[\sqrt{R}(p+q)]}{r\sec[\sqrt{R}(p+q)]+1}$$
$$\equiv -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} - \frac{p_x}{2}\sqrt{R}\tan\left[\sqrt{R}\frac{(p+q)-\varphi_3}{2}\right], \tag{21}$$

$$G_3 = H_{3y} = -\frac{R}{4}p_x q_y \sec^2\left[\sqrt{R}\frac{(p+q)-\varphi_3}{2}\right]. \tag{22}$$

In (21), we use the relation

$$\frac{\sqrt{1-r^2}\sec(\theta) - \tan(\theta)}{r\sec(\theta)+1} \equiv -\tan\left(\frac{\theta-\varphi_3}{2}\right),$$

where

$$\varphi_3 = \tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right).$$

$$H_4 = -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{\frac{1-r^2}{R}}\frac{R\csc[\sqrt{R}(p+q)]}{r\csc[\sqrt{R}(p+q)]+1} + \frac{p_x}{2}\frac{\sqrt{R}\cot[\sqrt{R}(p+q)]}{r\csc[\sqrt{R}(p+q)]+1}$$
$$\equiv -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{2}\sqrt{R}\cot\left[\sqrt{R}\frac{(p+q)+\varphi_4}{2}\right], \tag{23}$$

$$G_4 = H_{4y} = -\frac{R}{4}p_x q_y \csc^2\left[\sqrt{R}\frac{(p+q)+\varphi_4}{2}\right]. \tag{24}$$

In (23), we use the relation

$$\frac{\sqrt{1-r^2}\csc(\theta) + \cot(\theta)}{r\csc(\theta)+1} \equiv \cot\left(\frac{\theta+\varphi_4}{2}\right),$$

where

$$\varphi_4 = \tan^{-1}\left(\frac{r}{\sqrt{1-r^2}}\right).$$

According to the method mentioned above in Sect. 2 and (6) and (7), we assume that (13) has solution of the form $H(x,y,t) = a_0(x,y,t) + a_1(x,y,t)g(w)$, when $R = r = 0$; then we obtain the rational solutions

$$H_5 = -\frac{B(t)p_{xx} + p_t}{2B(t)p_x} + \frac{p_x}{p+q}, \tag{25}$$

$$G_5 = H_{5y} = -\frac{p_x q_y}{(p+q)^2}, \tag{26}$$

where p and q are arbitrary functions of $\{x,t\}$ and $\{y\}$, respectively.

By careful analysis, we find that when re-defining $p = \exp\{-\sqrt{R}p\}$, $q = \exp\{\sqrt{R}(q + \varphi_1)\}$ in solutions (25) and (26), solutions (17) and (18) can be obtained. Similarly, if taking $p = -\exp\{-\sqrt{R}p\}$, $q = \exp\{\sqrt{R}(q+\varphi_2)\}$ in solutions (25) and (26), solutions (19) and (20) can be recovered. When considering $p = \exp\{-\mathrm{i}\sqrt{R}p\}$, $q = \exp\{\mathrm{i}\sqrt{R}(q - \varphi_3)\}$ in solutions (25) and (26), solutions (21) and (22) can be obtained. If letting $p = -\exp\{-\mathrm{i}\sqrt{R}p\}$, $q = \exp\{\mathrm{i}\sqrt{R}(q + \varphi_4)\}$ in solutions (25) and (26), solutions (23) and (24) can be recovered. Therefore, only solutions (25) and (26) are essentially effective.

4 Soliton Fusion and Fission among Semifoldons

In this section, we will discuss some special types of interesting interaction behaviors for the quantity

$$U \equiv G_5 \equiv -\frac{p_x q_y}{(p+q)^2}. \tag{27}$$

Because of the arbitrariness of the functions $p(x,t)$ and $q(y)$, included in (27), the quantity U possesses quite rich structures. For instance, if we select the functions p and q appropriately, we can obtain many kinds of localized solutions, like the dromions, lumps, ring soliton and oscillated dromion, breathers solution, fractal-dromion and fractal-lump soliton structures [25]. In addition to the usual localized structures, some new localized excitations like peakon, compacton, folded solitary wave and foldon solutions of quantity U are found by selecting some types of lower-dimensional appropriate functions [17–21, 25]. The properties of peakon–peakon, dromion–dromion, compacton–compacton, and foldon-foldon interactions [17–21] can be re-discussed by quantity U. In [21], we investigate some novel semi-foldon structures and some special dromions and peakons both analytically and graphically. Since these similar situations have been widely discussed in some previous literatures [17–21, 25], the related plots are neglected in our present paper. From above brief discussions, one may conclude that the field U may possess some novel properties that have not been revealed until now. Recently, it has been reported both theoretically and experimentally that fission and fusion phenomena can happen for $(1+1)$-dimensional solitons or solitary waves [6]. Now we pay our attention to the soliton fusion and fission phenomena.

4.1 Fusion Phenomenon among Bell-Like and Compacton-Like Semi-Foldons

Besides the fusion phenomena found between usual localized coherent structures in [7], we also find some novel fusion phenomenon among multi-valued (folded) localized excitations, i.e. bell-like and compacton-like semi-foldons, when p and q possess the following forms

$$p = 22 + \exp(x+t) + \begin{cases} 0, & x - t \le -\frac{5\pi}{8}, \\ 16\sin(0.8x - 0.8t) + 16, & -\frac{5\pi}{8} < x - t \le \frac{5\pi}{8}, \\ 32, & x - t > \frac{5\pi}{8}, \end{cases} \tag{28}$$

$$q_y = -\operatorname{sech}^2(\zeta), \qquad x = \zeta - 1.5\tanh(\zeta), \qquad q = \int^{\zeta} q_y y_\zeta \, d\zeta. \tag{29}$$

We can observe the intriguing fusion phenomenon in Fig. 1. The big compacton-like semi-foldon travels along positive x-axis, and the small bell-like semi-foldon moves along negative x-axis, and ultimately they fuse into single bell-like semi-foldon running along negative x-axis stably even while we run program for rather long times ($t = 10^3$).

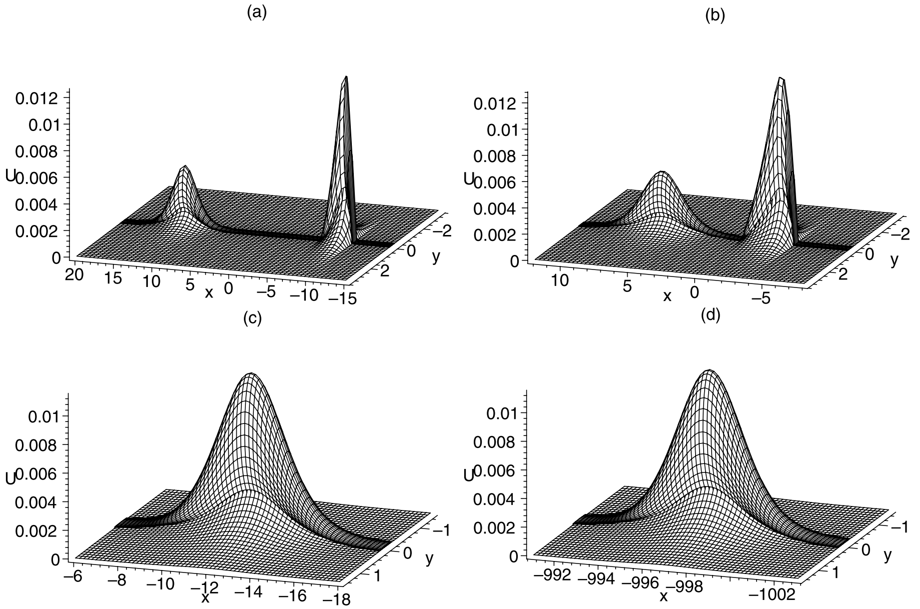

Fig. 1 The time evolutional profiles of bell-like and compacton-like semi-foldons fuse into single bell-like semi-foldon for the quantity U (27) with conditions (28) and (29) at different times (**a**) $t=-8$; (**b**) $t=-2$; (**c**) $t=15$; (**d**) $t=1000$

4.2 Fission Phenomenon of Bell-Like Semi-Foldon

If considering the arbitrary function p as

$$p = 12 + [\exp(5x+6t) + 0.8\exp(2x+3t) + 0.5\exp(2x+4t)][1+\exp(2x+3t)]^{-2}, \quad (30)$$

and q is taken as (29), then one can obtain a new kind of fission solitary wave solution for the expression U, which possesses apparently different evolutional property compared with fusion phenomena in Fig. 1. From Figs. 2(a–d), one can clearly see that single bell-like semi-foldon fissions into three bell-like semi-foldons (two semi-foldons and one anti-semifoldon) with different amplitudes. It is interesting to mention that these bell-like semi-foldons exhibit different evolutional behaviors, i.e., the big semi-foldon is almost static in position $\{x=2, y=0\}$, and the small semi-foldon and the anti-semifoldon travel along negative x-axis. They are stable and do not undergo additional fission at least not if running the program for longer periods of $t=65$.

5 Summary and Discussion

In conclusion, the general projective Riccati equation method is applied to obtain variable separation solutions of $(2+1)$-dimensional VCBK system. By further studying, we find that these variable separation solutions obtained by PREM, which seem independent, actually depend on each other. Based on the quantity (27) and choosing suitable functions p and q, new types of fusion and fission phenomena among peakon, compacton, dromion and semifoldon are firstly investigated. Due to the experimental realization of soliton fusion and

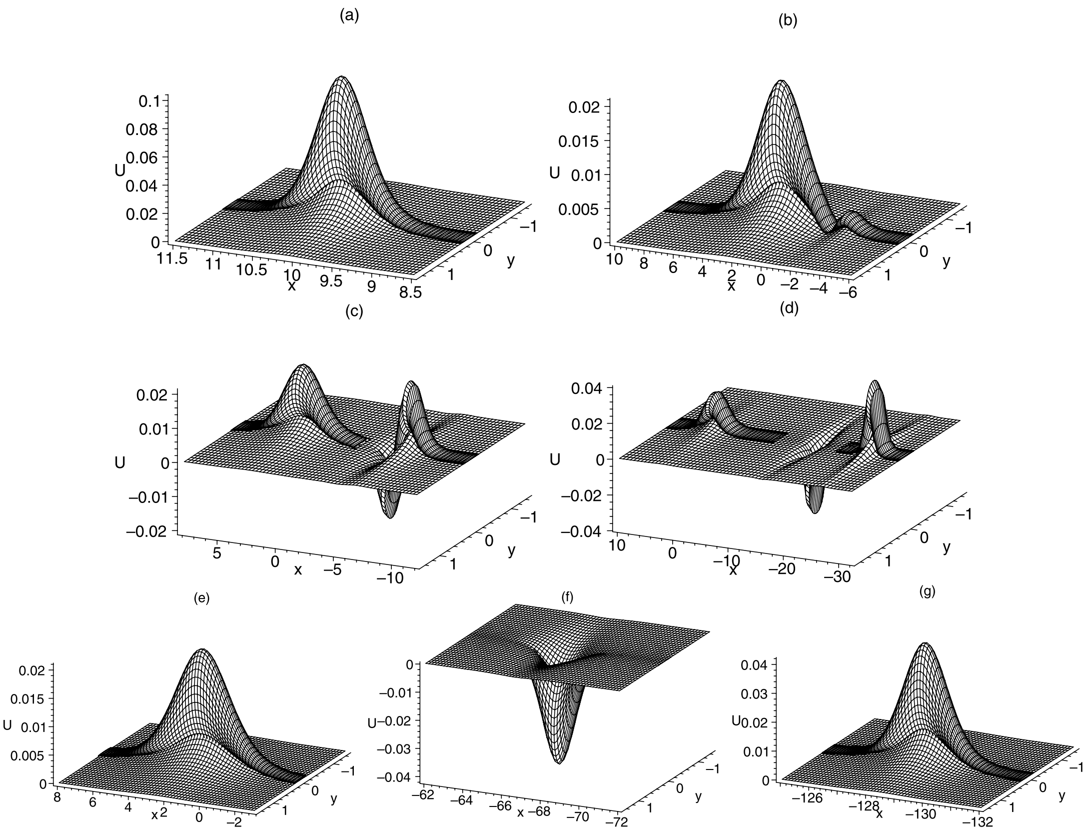

Fig. 2 Single bell-like semi-foldon fission into three bell-like semi-foldons: time evolutional profile for the quantity U (27) with conditions (29) and (30) at different times (**a**) $t=-8$; (**b**) $t=1$; (**c**) $t=4$; (**d**) $t=14$; (**e**), (**f**) and (**g**). The stable bell-like semi-foldons and anti-semi-foldon at time $t=65$, respectively

fission in many physical systems with some special conditions [2, 3], we think that the discussions about soliton fusion and fission phenomena in higher dimensional systems in this paper are significant and interesting. Clearly, there are some pending issues to be further studied. How to quantify the notion of soliton fusion and fission phenomena? What is the general equation for the distribution of the energy and momentum for these soliton fusion and fission behaviors? What are the necessary and sufficient conditions for soliton fission and soliton fusion which have been pointed in [4–6] in the $(1 + 1)$-dimensional cases?

What we have obtained is also verify that the general projective Riccati equation method is quite useful to generate abundant localized excitations. Besides these systems listed in this paper, we can also obtain the variable separation solutions of $(2 + 1)$-dimensional KdV equation, BLP system, DLW system, and the like. For the limit of length, we do not list them here. In our future work, we will devote to generalizing this method to the differential-difference equations and $(1 + 1)$-dimensional and $(3 + 1)$-dimensional nonlinear systems.

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